Evolution and global collapse of trapped Bose condensates under variations of the scattering length

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We develop the idea of selectively manipulating the condensate in a trapped Bose-condensed gas, without perturbing the thermal cloud. The idea is based on the possibility to modify the mean field interaction between atoms (scattering length) by nearly resonant incident light or by spatially uniform change of the trapping magnetic field. For the gas in the Thomas-Fermi regime we find analytical scaling solutions for the condensate wavefunction evolving under arbitrary variations of the scattering length a. The change of a from positive to negative induces a global collapse of the condensate, and the final stages of the collapse will be governed by intrinsic decay processes.

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The discovery of Bose-Einstein condensation (BEC) in trapped clouds of ultra-cold alkali atoms [1–3] opened unique possibilities to investigate collective many-body effects in dilute gases. Isolation of a trapped gas from the environment, provided by a wall-less magnetic confinement, makes this object attractive for studying fundamental problems of the physics of many-body quantum systems, such as relaxation and the loss of coherence in the evolving macroscopic quantum state (condensate). In ongoing experiments [4–8] the condensate is set into motion (for example, undergoes oscillations) by varying the confining field. Scaling theory of coherent evolution (without damping and relaxation) of a Bose condensate in harmonic traps under arbitrary frequency variations has been developed in [9,10]. It is important, however, that variations of the confining potential also cause the evolution of the thermal component of a trapped gas. Moreover, in the hydrodynamic regime the thermal cloud in many aspects evolves similarly to the condensate [9]. For example, the asymmetry of free expansion and eigenfrequencies of small oscillations are almost the same.

In this paper we develop the idea of selectively manipulating the condensate, without perturbing the thermal cloud. This is especially important for studying the interaction between the condensate and the thermal component. The idea is based on the possibility to modify the mean field interaction between atoms. At low temperatures the latter is proportional to the scattering length a which, as found in [11], can be changed under the influence of red-detuned nearly resonant light. Another option to modify the scattering length relies on the magnetic field dependence of a, predicted in [12], and assumes spatially uniform variations of the trapping field, without changing the trap frequencies. Since the shape of the condensate wavefunction is predetermined by the interaction between particles, the change of a will cause the evolution of the condensate at constant frequencies of the confining potential. Remarkably, at temperatures $T \gg n\tilde{U}$ (n is the gas density, $\tilde{U} = 4\pi\hbar^2 a/m$, and m the atom mass) only the condensate evolution will be pronounced, which resembles the picture of the fourth sound in superfluid helium. In this temperature range the shape of the thermal cloud is mainly determined by the trapping potential and temperature. The perturbation of the thermal component will be small compared to that of the condensate, at least as $n\tilde{U}/T$.

The change of a in the field of nearly resonant red-detuned light is provided by virtual radiative transitions of a pair of atoms to a bound electronically excited molecular state. Since the resonance dipole interaction between atoms in the excited state is much stronger than the interaction in the ground state, already for moderate light intensities the effective interatomic interaction and, hence, the scattering length can be significantly changed. It is even possible to switch the sign of a [11].

The magnetic field dependence of the scattering length, found in [12], has a resonance structure. Therefore, spatially uniform variations of the field can also change a substantially and reverse its sign.

Below we find analytical scaling solutions for the condensate wavefunction evolving under arbitrary variations of the scattering length. We consider the most interesting situation, where the mean field interaction between atoms greatly exceeds the level spacing in the trap (Thomas-Fermi regime). The case of switching a from positive to negative and inducing a global collapse of the condensate is specially analyzed.

Let us consider the evolution of a condensate in a harmonic potential $V(\mathbf{r}) = m\sum_i \omega_i^2 r_i^2/2$, with constant frequencies ω_i, induced by variations of the mean field interaction between particles. The equation for the condensate wavefunction $\Psi_0(\mathbf{r}, t)$, with above-condensate particles neglected, can be represented in the form

$$i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi_0 + \frac{m}{2}\sum_i \omega_i^2 r_i^2 \Psi_0 + \tilde{U}(t)|\Psi_0|^2\Psi_0. \quad (1)$$

Here $\tilde{U}(t) = 4\pi\hbar^2 a(t)/m$, and $a(t)$ is the time-dependent scattering length. Along the lines of the general method developed in [9], we introduce d scaling parameters $b_i(t)$, where d is the dimension of the system. Turning to new coordinate and time variables $\rho_i = r_i/b_i(t)$, $\tau(t) =$

$\int^t dt'\gamma(t')/\mathcal{V}(t')$, with $\gamma(t)=\tilde{U}(t)/\tilde{U}_0$ and $\tilde{U}_0$ the initial value of $\tilde{U}(t)$, we search for the solution in the form

$$\Psi_0(\mathbf{r},t)=\mathcal{V}^{-1/2}(t)\chi_0(\boldsymbol{\rho},\tau(t))\exp\left(i\Phi(\mathbf{r},t)\right), \qquad (2)$$

where the dimensionless volume $\mathcal{V}(t)=\prod_i b_i(t)$. Substituting Eq.(2) into Eq.(1) we require the cancellation of the $\nabla_\rho\chi_0$ terms. This leads to the relation for the phase:

$$\Phi(\mathbf{r},t)=(m/2\hbar)\sum_i r_i^2[\dot{b}_i(t)/b_i(t)]. \qquad (3)$$

The scaling parameters $b_i(t)$ will be chosen such that they are governed by the equations

$$\ddot{b}_i+\omega_i^2 b_i=\omega_i^2\gamma(t)/b_i\mathcal{V}(t), \qquad (4)$$

with initial conditions $b_i(0)=1$, $\dot{b}_i(0)=0$. Then, we arrive at the equation of motion

$$i\hbar\frac{\partial\chi_0}{\partial\tau}=-\frac{\hbar^2}{2m}\sum_i\frac{\mathcal{V}(t)}{\gamma(t)b_i^2(t)}\frac{\partial^2\chi_0}{\partial\rho_i^2}+\frac{m}{2}\sum_i\omega_i^2\rho_i^2\chi_0+\tilde{U}_0|\chi_0|^2\chi_0. \qquad (5)$$

In the case of initially repulsive ($a(0)>0$) and strong interaction between particles (Thomas-Fermi regime) the initial chemical potential $\mu_0=n_0\tilde{U}_0\gg\hbar\omega_i$ (n_0 is the maximum density in the initial static condensate), and the kinetic energy term in Eq.(5) is initially small compared to the non-linear interaction term . In the course of evolution the ratio of the kinetic to interaction term scales as $\varepsilon(t)=\sum_i(\hbar\omega_i/\mu_0 b_i(t))^2\mathcal{V}(t)/\gamma(t)$. Assuming that the condition $\varepsilon(t)\ll 1$ is satisfied at any t, the kinetic energy term can be omitted. In the case of a fast change of the interparticle interaction this, in particular, requires the inequality $|\gamma|\gg(\hbar\omega_i/\mu_0)^2$ at t close to 0. Then in the variables ρ_i, τ Eq.(5) is reduced to that for the static case, with initial interaction between particles. The solution has the well known form (see [13,14]):

$$\chi_0(\boldsymbol{\rho},\tau(t))=\frac{1}{\tilde{U}_0^{1/2}}\left(\mu_0-\frac{m}{2}\sum_i\omega_i^2\rho_i^2\right)^{1/2}\exp\left(\frac{-i\mu_0\tau(t)}{\hbar}\right) \qquad (6)$$

in the spatial region where the argument of the square root is positive and zero otherwise. Thus, the condensate evolution under arbitrary variations of the scattering length is described by a universal scaling solution for $\Psi_0(\mathbf{r},t)$, following from Eqs. (2) and (6). Actually, the problem is reduced to the solution of classical equations (4) for the scaling parameters $b_i(t)$ (cf. [9]).

We should emphasize that the results for $d=2$, obtained below, can be applied equally well to the radial evolution of Ψ_0 in very long samples where the axial frequency is much smaller than the radial one. This follows from the fact that then to a first approximation the dependence of Ψ_0 on the axial coordinate can be omitted.

In the limiting case of adiabatically slow variations of the scattering length, that is on a time scale $\tau_0\gg\omega_i^{-1}$, the solution of Eqs.(4) leads to simple relations

$$b_i(t)=[\gamma(t)]^{1/(2+d)}\equiv b(t); \qquad \mathcal{V}(t)=[\gamma(t)]^{d/(2+d)}. \qquad (7)$$

Assuming that at $t\gg\tau_0$ the scattering length acquires a constant value a_1 ($\tilde{U}(t)=\tilde{U}_1$, $\gamma(t)=\tilde{U}_1/\tilde{U}_0\equiv\gamma_1$), for the phase in Eq.(6) we obtain $\mu_0\tau(t)=\mu_1 t$, where the quantity $\mu_1=\mu_0\gamma_1^{2/(2+d)}$is equal to the chemical potential of the gas with the interparticle interaction $\tilde{U}_1$. Then, from Eqs. (6) and (2) one can see that the initial condensate is adiabatically transformed to the static condensate with the wavefunction corresponding to the interaction $\tilde{U}_1$.

For an abrupt change of the interparticle interaction from $\tilde{U}_0$ to $\tilde{U}_1$ scaling equations (4) take the form

$$\ddot{b}_i+\omega_i^2 b_i=\omega_i^2\gamma_1/b_i\mathcal{V}(t). \qquad (8)$$

In fact, for $\gamma_1=\tilde{U}_1/\tilde{U}_0>0$ these equations are similar to those for the scaling parameters of the condensate evolution after an abrupt change of the trapping potential, with initial frequencies being $\omega_i\sqrt{\gamma_1}$ and final frequencies ω_i (see [9]). The solution of Eqs.(8) gives oscillating functions $b_i(t)$ which ensure undamped oscillations of the condensate wavefunction. For γ_1 close to 1 the frequency spectrum of the condensate oscillations coincides with the set of eigenfrequencies of small shape oscillations of the initial condensate, found for cylindrically symmetric traps in the JILA [4] and MIT [7] experiments and calculated in [15,16,10,9]. It is important that an abrupt change of the scattering length is equivalent to equal relative change of all frequencies. This reduces the possibility of stochastization of the condensate evolution in an anisotropic harmonic potential, compared to the case of independent change of the frequencies.

The structure of the condensate oscillations caused by the change of the scattering length can be easily analyzed in the case of isotripic trapping configuration. For $d=2$ Eq.(8) has analytical solution

$$b_i^2(t)=b^2(t)=[1+\gamma_1+(1-\gamma_1)\cos 2\omega t]/2. \qquad (9)$$

Once the scattering length is decreased ($\gamma_1<1$) there will be "compression oscillations": Compression of the condensate will be followed by its expansion to the initial shape. An increase of a ($\gamma_1>1$) induces "expansion oscillations", i.e. the condensate is expanding and then compressing to the initial shape. In both cases the scaling parameter b varies from 1 to $\sqrt{\gamma_1}$. In the 3-d case the condensate oscillations are anharmonic, with a characteristic period somewhat smaller than that for $d=2$. The parameter b is varying from 1 to $\sqrt{2\gamma_1/3}$.

The dynamics of the system, induced by the change of the interparticle interaction, is drastically different from that caused by variations of the trap frequencies. In the former case the condensate oscillates, whereas the thermal component is only weakly perturbed and practically remains static. In the latter case the thermal cloud is fully involved in the oscillatory evolution process. Preferential evolution of the condensate under the change of

the scattering length resembles the fourth sound in superfluid helium, where the superfluid part of the density oscillates on the background of a static normal component. As well as in helium, the analysis of dynamic properties of a trapped gas in these conditions can be an effective method of identifying and studying BEC.

The change of the interaction between particles can lead to an interesting phenomenon, a global collapse of the condensate as a whole. Let us assume that initially the scattering length $a > 0$, and the system is in a stationary state. The sign of a can be switched to negative in the field of nearly resonant red-detuned light [11] or by spatially uniform variations of the confining magnetic field (see [12]). Then, for an abrupt change of a the solution of scaling equations (8) immediately leads to a self-compression (collapse) of the condensate. In the $d = 2$ isotropic case this directly follows from the exact solution (9): For $\gamma_1 < 0$ the scaling parameter b decreases and reaches zero at $t = t_*$, where

$$t_* = \omega^{-1} \arcsin\left(1/\sqrt{1+|\gamma_1|}\right). \quad (10)$$

In the 3-d isotropic case t_* is close to that from Eq.(10). In the vicinity of t_*, where $\omega\Delta t \ll \min[|\gamma_1|^{1/d}, |\gamma_1|^{-1/2}]$ ($\Delta t = t_* - t$), the solution of Eq.(8) takes the form

$$b^2(t) = [(\sqrt{|\gamma_1|}\omega\Delta t)(2+d)/\sqrt{2d}]^{4/(2+d)}. \quad (11)$$

In both cases the compression rate increases with decreasing Δt: The quantity $\dot{b}(t)/b(t) = 2/[(d+2)\Delta t]$, and the characteristic velocity of the condensate boundary, $v \propto [\Delta t]^{-d/(d+2)}$. In the course of collapse the small parameter $\varepsilon(t)$ remains constant ($d = 2$) or even decreases ($d = 3$), which justifies the neglect of the kinetic energy term in Eq.(5). The kinetic energy of the system is determined by the behavior of the phase $\Phi(\mathbf{r})$ (3) which drastically increases with decreasing Δt. A strong rise of the kinetic energy in the course of collapse is compensated by decreasing potential energy ($a < 0$), which ensures the conservation of the total energy.

The increase of density in the collapsing condensate enhances intrinsic inelastic processes, such as three-body recombination and spin-dipole relaxation in binary collisions. This leads to particle losses, since fast atoms and molecules produced in the inelastic processes escape from the trap. We will confine ourselves to the analysis of the influence of inelastic processes on the dynamics of collapse in the isotropic $d = 2$ and $d = 3$ cases. The atom loss rate is determined by the relation

$$\dot{N} = -\alpha_s \int d^d r |\Psi_0(\mathbf{r},t)|^4 - \alpha_r \int d^d r |\Psi_0(\mathbf{r},t)|^6, \quad (12)$$

where α_s and α_r are the rate constants of spin relaxation and three-body recombination, respectively. With increasing density in the course of collapse, the ratio of the recombination to relaxation rate increases. In ultracold alkali atom gases α_s is in the range $10^{-14} - 10^{-16}$ cm^3/s and $\alpha_r \sim 10^{-28}$ cm^6/s. Accordingly, considering initial condensate densities $n_0 \sim 10^{13} - 10^{14}$ cm^{-3} (currently achieved in Rb and Na BEC experiments [4–8]), we see that already for moderate compression of the condensate three-body recombination dominates over spin relaxation. Therefore, the process of spin relaxation is omitted in the further analysis. For the same reason, in the case of light-induced change of the scattering length we omit the process of photoassociation in pair collisions.

The loss of particles from the condensate influences the evolution of the condensate wavefunction. The character of the evolution depends on the density of the collapsing condensate, $n(t)$, and is predetermined by the relation between the characteristic time of three-body recombination, $\tau_r \sim (\alpha_r n^2)^{-1}$, and the correlation time $\tau_c \sim \hbar/n|\tilde{U}_1|$. The latter is responsible for establishing the shape of the condensate wavefunction corresponding to the instantaneous value of the number of particles $N(t)$. There will be two different stages of the evolution. The role of three-body recombination becomes especially important at densities

$$n \gtrsim n_* \equiv |\tilde{U}_1|/\hbar\alpha_r \gg n_0, \quad (13)$$

where the compression is very strong and $\tau_r \lesssim \tau_c$. Then already the shape of $|\Psi_0(\mathbf{r},t)|$ is determined by the recombination losses (see below). With realistic numbers $|a_1| \sim 10$ Å, and $\alpha_r \sim 10^{-28}$ cm^6/s, we see that the density n_* will be in the range $10^{17} - 10^{18}$ cm^{-3} and still satisfies the criterion of weakly interacting gas, $n_*|a|^3 \ll 1$.

In the density range $n \ll n_*$ the recombination time $\tau_r \gg \tau_c$ and the loss of particles occurs in a quasistationary regime. For $\Psi_0(\mathbf{r},t)$ one can use the scaling solution following from Eqs. (2), (6), with instantaneous values of N and the chemical potential μ. Since $\mu \sim N^{2/(2+d)}$, for the time dependence of the maximum condensate density in the quasistationary stage we obtain

$$n(t) = n_0[b(t)]^{-d}[N(t)/N_0]^{2/(2+d)}, \quad (14)$$

where N_0 is the initial number of condensate particles.

Below we assume that $|\gamma_1| \lesssim 1$. Then the characteristic time of collapse, $t_* \sim \omega^{-1}$. Since for realistic parameters of the system the initial recombination time $\tau_{r0} \gg \omega^{-1}$, a major part of the compression occurs without particle losses. This is the case even at t rather close to t_*, the condensate density being determined by Eq.(14) with the scaling parameter b from Eq.(11) and $N \approx N_0$:

$$n(t) \approx n_0(\sqrt{|\gamma_1|}\omega\Delta t)^{-2d/(2+d)}. \quad (15)$$

Eq.(15) is no longer valid when the number of lost particles becomes comparable with N_0. The characteristic time Δt_L and the density n_L at which this happens can be found from Eq.(12), with Ψ_0 determined by Eqs. (2), (5) and the scaling parameter $b(t)$ given by Eq.(11):

$$\dot{N} = -\frac{\alpha_r}{\tilde{U}_0^3}\int \frac{d^d\rho}{b^{2d}(t)}(\mu_0 - m\omega^2\rho^2/2)^3 \approx -\frac{N_0}{\tau_{r0}(\sqrt{|\gamma_1|}\omega\Delta t)^{4d/(2+d)}}.$$

Comparing the number of lost atoms with N_0 we obtain

$$\sqrt{|\gamma_1|}\omega\Delta t_L \approx (\sqrt{|\gamma_1|}\omega\tau_{r0})^{-(2+d)/(3d-2)}; \quad (16)$$

$$n_L \approx n_0(\sqrt{|\gamma_1|}\omega\tau_{r0})^{2d/(3d-2)}. \quad (17)$$

Assuming $\sqrt{|\gamma_1|}\omega\tau_{r0} \gg 1$ we see that $n_L \gg n_0$.

Eq.(13) can be rewritten as $n_* \approx n_0|\gamma_1|(\omega\tau_{r0})(\mu_0/\hbar\omega)$, and the condition $|\gamma_1| \gg (\hbar\omega/\mu_0)^2$ (see above) ensures the inequality $n_L \ll n_*$ which justifies the use of the quasistationary approach in deriving Eqs. (15), (16) and (17).

The condition $n_L \ll n_*$ means that the collapse continues to occur in the quasistationary regime also at times $\Delta t < \Delta t_L$. In this time interval, using Eqs. (11), (14), (16), and Eq.(12) written in the form $\dot{N} \approx -\alpha_r n^2(t)N(t)$, for the number of particles we find

$$N(t) \approx N_0(\Delta t/\Delta t_L)^{(3d-2)/4}. \quad (18)$$

The condensate density increases as

$$n(t) \approx n_0\sqrt{\tau_{r0}/\Delta t}, \quad (19)$$

i.e. slower than in the initial stage of the compression.

At times very close to t_*, i.e., at $\Delta t \approx \tau_{r*} = (\alpha_r n_*^2)^{-1}$, the density approaches n_*, and the picture of collapse changes. If n becomes larger than n_*, then the decrease of n due to particle losses already dominates over the dynamic compression. On the other hand, for $n < n_*$ the dynamic compression dominates and the density rises. Therefore, n_* proves to be a critical density value which is approximately conserved in the final stage of collapse. The decrease of the number of particles is determined by the relation $\dot{N} = -N/\tau_{r*}$, and we obtain

$$N(t) \approx N_* \exp\{-(t-t_*)/\tau_{r*}\}, \quad (20)$$

where $N_* \ll N_0$ and is determined by Eqs. (18) and (16), with $\Delta t \approx \tau_{r*}$. The characteristic size of the condensate decreases as $N(t)/n_*$.

The above described picture of collapse is somewhat idealized. For example, the mean interparticle separation at limiting density, $n_*^{-1/3}$, can become comparable with the characteristic radius of interatomic interaction. Nevertheless, we believe that the main results should remain unchanged. The scenario of collapse can be observed by measuring the recombination losses of particles.

Considering the global collapse in an anisotropic harmonic potential we should return to scaling equations (8), with $\gamma_1 < 0$. The rate of the self-compression is the highest in the direction corresponding to the largest frequency. In the case of cylindrical symmetry with the ratio of the axial to radial frequency, $\beta = \omega_z/\omega_\rho < 1$, there will be a characteristic time t_* at which $b_\rho(t_*) = 0$, but $b_z(t_*)$ remaines finite. In the time interval, where $b_\rho(t) \ll 1$, Eq.(8) for $b_\rho(t)$ is reduced to that for the isotropic 2-d collapse, with γ_1 replaced by $\gamma_1/b_z(t_*)$. Thus, the global collapse will occur along the lines of the above described 2-d scenario, in combination with a "slow" compression in the axial direction. For $\beta > 1$ the condensate predominantly collapses in the axial direction and, due to a quasi-one-dimensional character of the self-compression, the kinetic energy term in Eq.(5) increases. Eventually the parameter $\varepsilon(t)$ becomes of order unity, and our scaling approach breaks down. Such a rise of the kinetic energy prevents the further compression and can lead to a non-trivial oscillatory character of the global collapse.

Density fluctuations on the background of the global collapse can lead to the appearence of local collapses, with linear dimensions of order the healing length. Although their influence on the global collapse can be reduced by decreasing the ratio $n_0\tilde{U}/\hbar\omega$, the study of the role of the local collapses requires a separate analysis.

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